Department of Electrical and Computer Engineering
University of Massachusetts/Amherst
ECE 645: Digital Communications Fall 2011
Homework #1 Solutions

Problem 4.4 :

$$\begin{aligned} E\left[x(t+\tau)x(t)\right] &= A^2 E\left[\sin\left(2\pi f_c(t+\tau)+\theta\right)\sin\left(2\pi f_c t+\theta\right)\right] \\ &= \frac{A^2}{2}\cos 2\pi f_c\tau - \frac{A^2}{2}E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] \end{aligned}$$

where the last equality follows from the trigonometric identity :
$\sin A \sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$. But :

$$\begin{aligned} E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] &= \int_0^{2\pi}\cos\left(2\pi f_c(2t+\tau)+2\theta\right)p(\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos\left(2\pi f_c(2t+\tau)+2\theta\right)d\theta = 0 \end{aligned}$$

Hence :

$$E\left[x(t+\tau)x(t)\right] = \frac{A^2}{2}\cos 2\pi f_c\tau$$

Problem 4.11 :

(a) As an orthonormal set of basis functions we consider the set

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad f_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

(b) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

(c) The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\left[\begin{array}{cccc} 4 & -2 & -2 & -1 \end{array}\right]\right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned}
d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\left[\begin{array}{cccc} 1 & 0 & -2 & 0 \end{array}\right]\right|^2} = \sqrt{5} \\
d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\left[\begin{array}{cccc} 1 & 1 & 1 & -3 \end{array}\right]\right|^2} = \sqrt{12} \\
d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\left[\begin{array}{cccc} -3 & 2 & 0 & 1 \end{array}\right]\right|^2} = \sqrt{14} \\
d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\left[\begin{array}{cccc} -3 & 3 & 3 & -2 \end{array}\right]\right|^2} = \sqrt{31} \\
d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\left[\begin{array}{cccc} 0 & 1 & 3 & -3 \end{array}\right]\right|^2} = \sqrt{19}
\end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 5.2 :

(a) The impulse response of the matched filter is :

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(b) The output of the matched filter at $t = T$ is :

$$\begin{aligned}
g(T) &= h(t) \star s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2 \cos^2(2\pi f_c(T-\tau))d\tau \\
&\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2 \cos^2(2\pi f_c v)dv \\
&= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4 \times 2\pi f_c} - \frac{1}{8 \times (2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4 \times 2\pi f_c} - \frac{1}{8 \times (2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}$$

(c) The output of the correlator at $t = T$ is :

$$\begin{aligned}
q(T) &= \int_0^T s^2(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T \tau^2 \cos^2(2\pi f_c \tau)d\tau
\end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

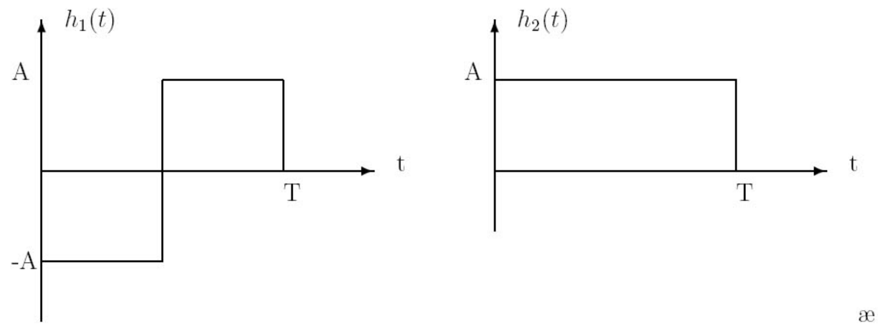
Problem 5.8 :

(a) Since the given waveforms are the equivalent lowpass signals :

$$\mathcal{E}_1 = \frac{1}{2}\int_0^T |s_1(t)|^2\, dt = \frac{1}{2}A^2 \int_0^T dt = A^2T/2$$
$$\mathcal{E}_2 = \frac{1}{2}\int_0^T |s_2(t)|^2\, dt = \frac{1}{2}A^2 \int_0^T dt = A^2T/2$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also :$\rho_{12} = \frac{1}{2\mathcal{E}}\int_0^T s_1(t)s_2^*(t)dt = 0.$

(b) Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T-t)$. The following figure shows $h_i(t)$:



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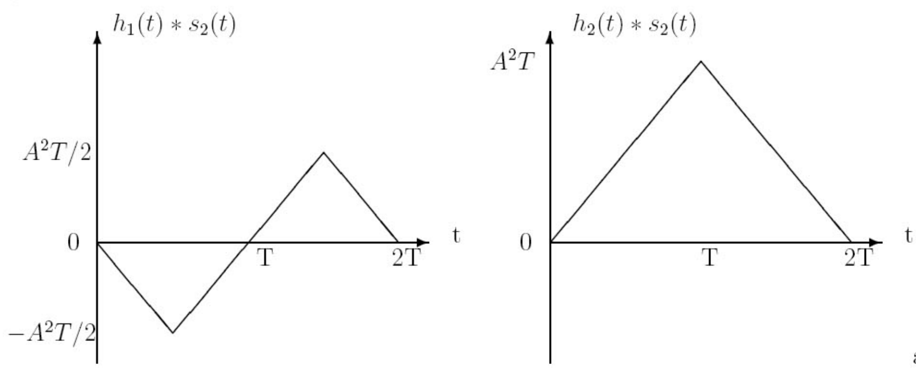
(c)



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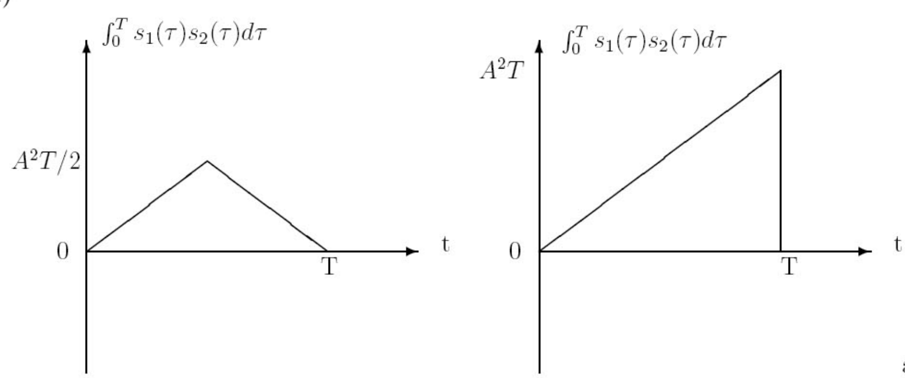
(d)



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(e) The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

(f) Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.

Problem 5.19 :

(a) The PDF of the noise n is :

$$p(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

where $\lambda = \frac{\sqrt{2}}{\sigma}$ The optimal receiver uses the criterion :

$$\frac{p(r|A)}{p(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1 \Longrightarrow r \underset{-A}{\overset{A}{\gtrless}} 0$$

The average probability of error is :

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\ &= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\ &= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\ &= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\ &= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{\frac{-\sqrt{2}A}{\sigma}} \end{aligned}$$

(b) The variance of the noise is :

$$\begin{aligned} \sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2 dx \\ &= \lambda\int_{0}^{\infty} e^{-\lambda x}x^2 dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2} = \sigma^2 \end{aligned}$$

Hence, the SNR is:

$$\text{SNR} = \frac{A^2}{\sigma^2}$$

and the probability of error is given by:

$$P(e) = \frac{1}{2}e^{-\sqrt{2SNR}}$$

For $P(e) = 10^{-5}$ we obtain:

$$\ln(2 \times 10^{-5}) = -\sqrt{2SNR} \Longrightarrow SNR = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then the probability of error for antipodal signalling is:

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{SNR}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{SNR} = 4.26$ and therefore SNR $= 18.1476 = 12.594$ dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 5.27 :

Using the Pythagorean theorem for the four-phase constellation, we find:

$$r_1^2 + r_1^2 = d^2 \Longrightarrow r_1 = \frac{d}{\sqrt{2}}$$

The radius of the 8-PSK constellation is found using the cosine rule. Thus:

$$d^2 = r_2^2 + r_2^2 - 2r_2^2 \cos(45^o) \Longrightarrow r_2 = \frac{d}{\sqrt{2-\sqrt{2}}}$$

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by:

$$P_{4,av} = \frac{d^2}{2}, \qquad P_{8,av} = \frac{d^2}{2-\sqrt{2}}$$

Thus, the additional transmitted power needed by the 8-PSK signal is:

$$P = 10\log_{10} \frac{2d^2}{(2-\sqrt{2})d^2} = 5.3329 \text{ dB}$$

We obtain the same results if we use the probability of error given by (see 5-2-61) :

$$P_M = 2Q\left[\sqrt{2\gamma_s} \sin\frac{\pi}{M}\right]$$

where γ_s is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

$$\gamma_{4,s} \sin^2\frac{\pi}{4} = \gamma_{8,s} \sin^2\frac{\pi}{8} \Longrightarrow 10\log_{10}\frac{\gamma_{8,s}}{\gamma_{4,s}} = 20\log_{10}\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}} = 5.3329 \text{ dB}$$

Since we consider that error ocuur only between adjacent points, the above result is equal to the additional transmitted power we need for the 8-PSK scheme to achieve the same distance d between adjacent points.