Question 1)

The fourier transform of a signal is

$$X(e^{jw}) = \frac{1}{1-\alpha e^{-jw}} \qquad -1 < \alpha < 0$$

ⓐ Find & sketch $Re\{X(e^{jw})\}$? ⓑ Find & sketch $Im\{X(e^{jw})\}$?

$$\boxed{e^{-jw} = \cos(w) - j\sin(w)}$$

$$X(e^{jw}) = \frac{1}{1-\alpha(\cos(w) - j\sin(w))}$$

$$X(e^{jw}) = \frac{1}{1-\alpha\cos(w) + j\alpha\sin(w)}$$

METHOD ① :

→ To remove the imaginary part from denominator

$$X(e^{jw}) = \frac{1}{1-\alpha\cos(w)+j\alpha\sin(w)} \cdot \frac{1-\alpha\cos(w)-j\alpha\sin(w)}{1-\alpha\cos(w)-j\alpha\sin(w)}$$

$$X(e^{jw}) = \frac{1-\alpha\cos(w)-j\alpha\sin(w)}{1-\alpha\cos(w)+\cancel{j\alpha\sin(w)}-\alpha\cos(w)+\alpha^2\cos^2(w)-\cancel{j\alpha^2\cos(w)\sin(w)}-\cancel{j\alpha\sin(w)}+\cancel{j\alpha^2\cos(w)\sin(w)}-j^2\alpha^2\sin^2(w)}$$

$$X(e^{jw}) = \frac{1-\alpha\cos(w)-j\alpha\sin(w)}{1-2\alpha\cos(w)+\alpha^2\cos^2(w)-j^2\alpha^2\sin^2(w)}$$

$$\boxed{j^2 = -1}$$

$$X(e^{jw}) = \frac{1-\alpha\cos(w)-j\sin(w)}{1-2\alpha\cos(w)+\alpha^2(\cos^2(w)+\sin^2(w))}$$

$$\boxed{\sin^2(x)+\cos^2(x) = 1}$$

$$X(e^{jw}) = \frac{1-\alpha\cos(w) - j\sin(w)}{\underbrace{1-2\alpha\cos(w)+\alpha^2}_{\text{No imaginary part}}}$$

→ Now we can split $X(e^{jw})$ into real and imaginary part.

$$X(e^{jw}) = \underbrace{\frac{1-\alpha\cos(w)}{1-2\alpha\cos(w)+\alpha^2}}_{\text{Real Part}} - j\underbrace{\frac{\sin(w)}{1-2\alpha\cos(w)+\alpha^2}}_{\text{Imaginary Part}}$$

METHOD 2:

→ Direct formula method.

$$\boxed{Re\{z\} = \frac{1}{2}(z + \bar{z})}$$

$$Re\{X(e^{jw})\} = \frac{1}{2}\left(X(e^{jw}) + X(e^{jw})^*\right)$$

$$Re\{X(e^{jw})\} = \frac{1}{2}\left[\frac{1}{1-\alpha e^{-jw}} + \frac{1}{1-\alpha e^{jw}}\right]$$

$$Re\{X(ejw)\} = \frac{1}{2} \cdot \frac{1-\alpha e^{+jw} + 1 - \alpha e^{-jw}}{(1-\alpha ejw)(1-\alpha e^{-jw})}$$

$$Re\{X(e^{jw})\} = \frac{1}{2} \cdot \frac{1 - \alpha\cos(w) - \alpha \cancel{j\sin(w)} + 1 - \alpha\cos(w) + \alpha \cancel{j\sin(w)}}{1 - \alpha e^{-jw} - \alpha e^{jw} + \alpha^2 e^{jw}.e^{-jw}}$$

$$Re\{X(ejw)\} = \frac{1}{2} \cdot \frac{2 - 2\alpha\cos(w)}{1 - \alpha\left(\cos(w) - j\sin(w) + \cos(w) + j\sin(w)\right) + \alpha^2\left(e^{jw-jw}\right)}$$

$$Re\{X(e^{jw})\} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\left(1-\alpha\cos(w)\right)}{1 - 2\alpha\cos(w) + \alpha^2}$$

$$\boxed{Re\{X(e^{jw})\} = \frac{1-\alpha\cos(w)}{1-2\alpha\cos(w)+\alpha^2}}$$

$$\boxed{Im\{z\} = \frac{1}{2}(z - \bar{z})}$$

$$Im\{X(e^{jw})\} = \frac{1}{2j}\left(X(e^{jw}) - X^*(e^{jw})\right)$$

$$Im\{X(e^{jw})\} = \frac{1}{2j} \cdot \left[\frac{1}{1-\alpha e^{-jw}} - \frac{1}{1-\alpha e^{jw}}\right]$$

$$Im\{X(e^{jw})\} = \frac{1}{2j} \cdot \left[\frac{\cancel{1} - \alpha e^{jw} - \cancel{1} + \alpha e^{-jw}}{(1-\alpha e^{-jw})(1-\alpha e^{jw})}\right]$$

$$Im\{X(e^{jw})\} = \frac{1}{2j} \cdot \frac{-\alpha \cancel{\cos(w)} - j\alpha\sin(w) + \alpha\cancel{\cos(w)} - j\alpha\sin(w)}{1 - 2\alpha\cos(w) + \alpha^2}$$

$$\boxed{Im\{X(e^{jw})\} = \frac{-\sin(w)}{1-2\alpha\cos(w)+\alpha^2}}$$

(c) Find & Sketch $|X(e^{j\omega})|$?

$$\boxed{|X(e^{j\omega})| = \sqrt{X(e^{j\omega})\, X^{*}(e^{j\omega})}}$$

$$|X(e^{j\omega})| = \sqrt{\frac{1}{1-\alpha e^{-j\omega}} \cdot \frac{1}{1-\alpha e^{j\omega}}}$$

$$|X(e^{j\omega})| = \sqrt{\frac{1}{1-\alpha e^{j\omega}-\alpha e^{-j\omega}+\alpha^{2} e^{-j\omega}\cdot e^{j\omega}}}$$

$$|X(e^{j\omega})| = \left[\frac{1}{1-2\alpha e^{j\omega}-\alpha e^{-j\omega}+\alpha^{2}}\right]^{1/2}$$

$$|X(e^{j\omega})| = \left[\frac{1}{1-\alpha\left(\cos(\omega)+j\sin(\omega)+\cos(\omega)-j\sin(\omega)\right)+\alpha^{2}}\right]^{1/2}$$

$$|X(e^{j\omega})| = \left[\frac{1}{1-2\alpha\cos(\omega)+\alpha^{2}}\right]^{1/2} = \sqrt{\frac{1}{1-\alpha e^{j\omega}-\alpha e^{-j\omega}+\alpha^{2}}}$$

POSSIBLE SUBSTITUTION

$$\boxed{\frac{e^{j\omega}-e^{-j\omega}}{2} = \sin(\omega)}$$

POSSIBLE SUBSTITUTION

$$\boxed{\frac{e^{j\omega}+e^{-j\omega}}{2} = \cos(\omega)}$$

(d) Find & Sketch $\angle X(e^{j\omega})$?

$$\boxed{\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{\text{Img}\{X(e^{j\omega})\}}{\text{Real}\{X(e^{j\omega})\}}\right)}$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{\dfrac{\sin(\omega)}{\cancel{1-2\alpha\cos(\omega)+\alpha^{2}}}}{\dfrac{1-\alpha\cos(\omega)}{\cancel{1-2\alpha\cos(\omega)+\alpha^{2}}}}\right) = \tan^{-1}\left(\frac{-\sin(\omega)}{1-\alpha\cos(\omega)}\right)$$

(e) $h[n] = b^n u[n]$ find the output $y[n]$ and the difference equation describing the system?

→ Convolution in time domain is multiplication in frequency domain.

$$H(e^{jw}) = \frac{1}{1-be^{-jw}} \quad \text{assuming} \quad |b|<1$$

$$\therefore \quad Y(e^{jw}) = H(e^{jw}) \, . \, X(e^{jw})$$

$$Y(e^{jw}) = \frac{1}{1-be^{-jw}} \cdot \frac{1}{1-ae^{-jw}} = \frac{1}{(1-be^{-jw})(1-ae^{-jw})}$$

→ Using partial Equations to split up the $Y(e^{jw})$

$$Y(e^{jw}) = \frac{1}{(1-be^{-jw})(1-ae^{-jw})} = \frac{A}{(1-be^{-jw})} + \frac{B}{(1-ae^{-jw})}$$

$$A(1-ae^{-jw}) + B(1-be^{-jw}) = 1$$

$$B\left(1-be^{+\ln(\frac{1}{a})}\right) = 1$$

$$B\left(1-\frac{b}{a}\right) = 1$$

$$\boxed{B = \frac{1}{\frac{a-b}{a}} = \frac{a}{a-b}}$$

$$A\left(1-ae^{+\ln(\frac{1}{b})}\right) = 1$$

$$A\left(1-\frac{a}{b}\right) = 1$$

$$A = \frac{1}{\frac{b-a}{b}}$$

$$\boxed{A = \frac{b}{b-a}}$$

$1-ae^{-jw} = 0$

$ae^{-jw} = 1$

$a = \frac{1}{e^{-jw}}$

$e^{-jw} = \frac{1}{a}$

$-jw = \ln\left(\frac{1}{a}\right)$

$w = \frac{\ln\left(\frac{1}{a}\right)}{-j}$

$w = j\ln\left(\frac{1}{a}\right)$

$1-be^{-jw} = 0$

$be^{-jw} = 1$

$e^{-jw} = \frac{1}{b}$

$w = +j\ln\left(\frac{1}{b}\right)$

$$Y(e^{jw}) = \left(\frac{b}{b-a}\right) \cdot \frac{1}{1-be^{-jw}} + \left(\frac{a}{a-b}\right) \cdot \frac{1}{1-ae^{-jw}}$$

$$\boxed{\therefore \quad y[n] = \left[\left(\frac{b}{b-a}\right) b^n + \left(\frac{a}{a-b}\right) a^n\right] \cdot u[n]}$$

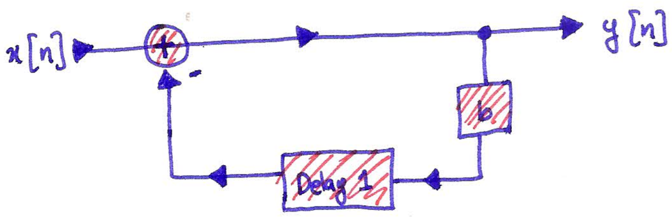
→ Finding the difference Equation.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - be^{-j\omega}}$$

$$Y(e^{j\omega}) - Y(e^{j\omega}).b.e^{-j\omega} = X(e^{j\omega})$$

IFT

$$\boxed{y[n] - b\,y[n-1] = x[n]}$$



Question 2) Let $X(e^{j\omega})$ denote the Fourier transform of $x[n] = \{-1,0,1,\underline{\underline{2}},1,0,1,2,1,0,-1\}$

Find $|X(e^{j\omega})|$ when $\omega = 0$?

$$x[n] = -\delta[n+3] + \delta[n+1] + 2\delta[n] + 1\delta[n-1] + 1\delta[n-3] + 2\delta[n-4] + 1\delta[n-5] - 1\delta[n-7]$$

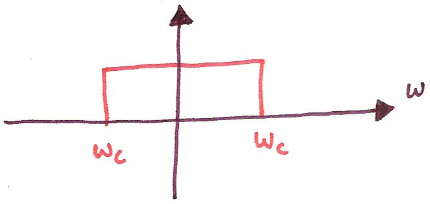
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

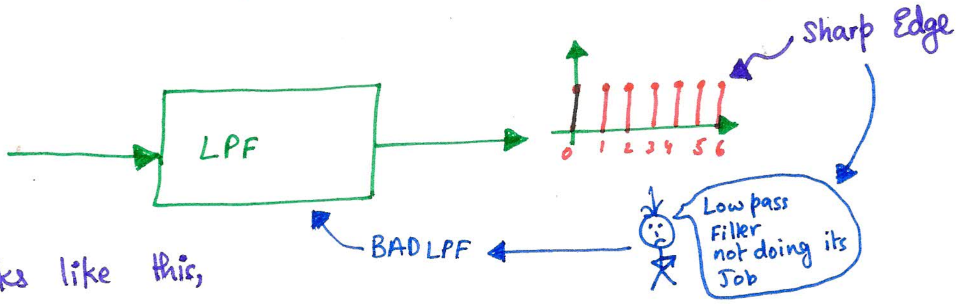
$$X(e^{j0}) = \sum_{n=-\infty}^{n=+\infty} x[n]\, e^{0} = \sum_{n=-3}^{7} x[n] = -1 + 1 + 2 + 1 + 1 + 2 + 1 - 1$$

$$\boxed{X(e^{j0}) = 6}$$

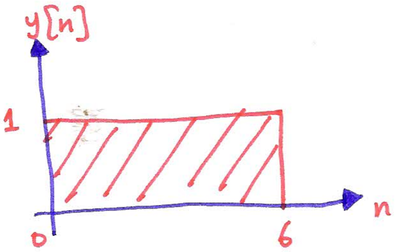
$$\boxed{|X(e^{j\omega})| \text{ when } \omega = 0 = 6}$$

Question 3) An ideal low pass filter $H(e^{j\omega})$ with cut-off frequency ω_c, if the output for some input is given by $y[n] = \begin{cases} 1 & 0 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$. What is the only possible value for ω_c?

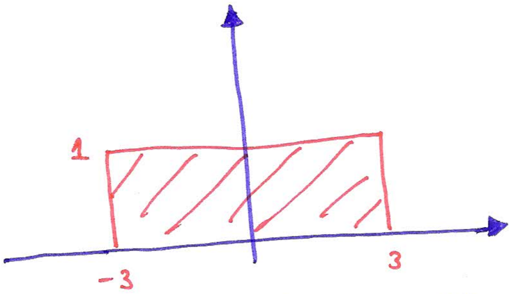




So the output looks like this,



If we take signal

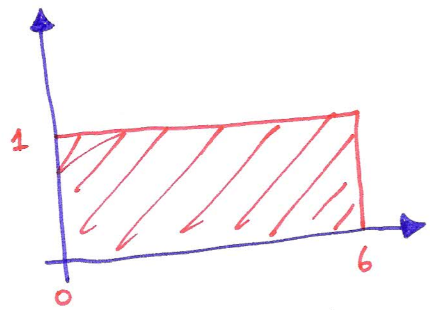


It can be represented as

$$x[n] = \begin{cases} 1 & |n| \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$X(e^{j\omega}) = \frac{\sin\left(\omega\left(3+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Shifting the signal three n's to right yields



It can be represented as

$$y[n] = \begin{cases} 1 & 0 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

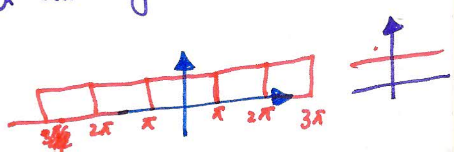
3 times Shift

$$Y(e^{j\omega}) = \frac{\sin\left(\omega\left(3+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)} \cdot e^{-j3\omega}$$

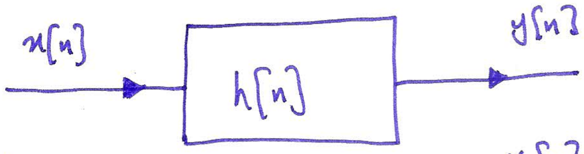
$$\boxed{\therefore Y(e^{j\omega}) = \frac{\sin(3.5\omega)}{\sin\left(\frac{\omega}{2}\right)} \cdot e^{-j3\omega}}$$

(Square Wave)

The low pass filter is allowing very abrupt changes in amplitudes at the output, which means that the low pass filter is not doing its job and allowing all frequencies to pass by. Now this is only possible if $\omega_c = \pi$.

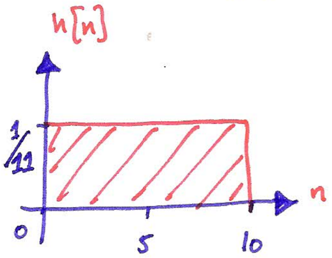


Question 4) What is the frequency response $H(e^{j\omega})$ of the 11-point moving averager given by $y[n] = \frac{1}{11}\sum\limits_{k=0}^{10} x[n-k]$?



if $x[n] = \delta[n]$ $\quad$ $y[n] = h[n]$

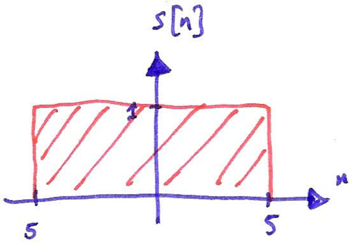
$$\therefore \; h[n] = \frac{1}{11}\sum_{k=0}^{10} \delta[n-k]$$



$$h[n] = \begin{cases} 1/11 & 0 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = \; ?$$

let me take the following signal

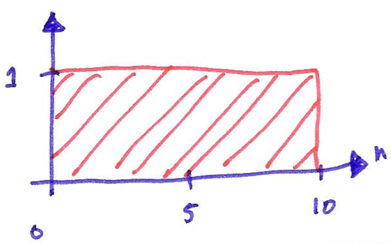


then,

$$s[n] = \begin{cases} 1 & |n| \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$$S(e^{j\omega}) = \frac{\sin\left(\omega \cdot \frac{11}{2}\right)}{\sin(\omega/2)}$$

let me shift this signal 5 units to right



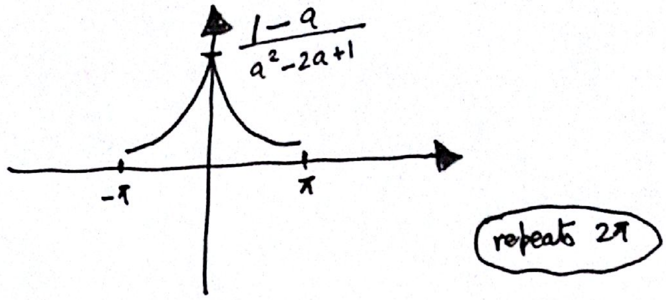
then,

$$s[n] = \begin{cases} 1 & 0 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

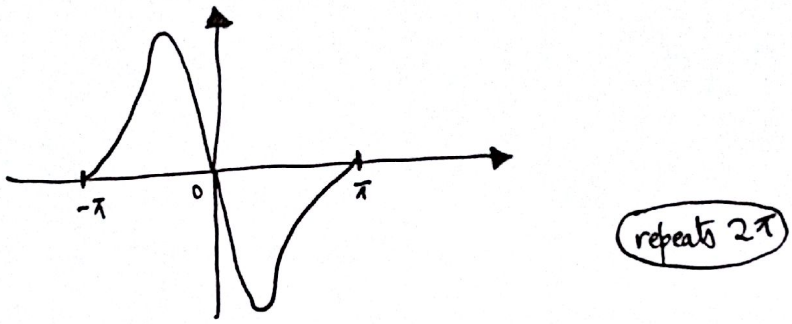
$$S(e^{j\omega}) = \frac{\sin\left(\omega \frac{11}{2}\right)}{\sin(\omega/2)} \cdot e^{-j5\omega}$$

$$\boxed{\therefore \; H(e^{j\omega}) = \frac{1}{11} \cdot \frac{\sin\left(\omega \frac{11}{2}\right)}{\sin(\omega/2)} \cdot e^{-5j\omega}}$$

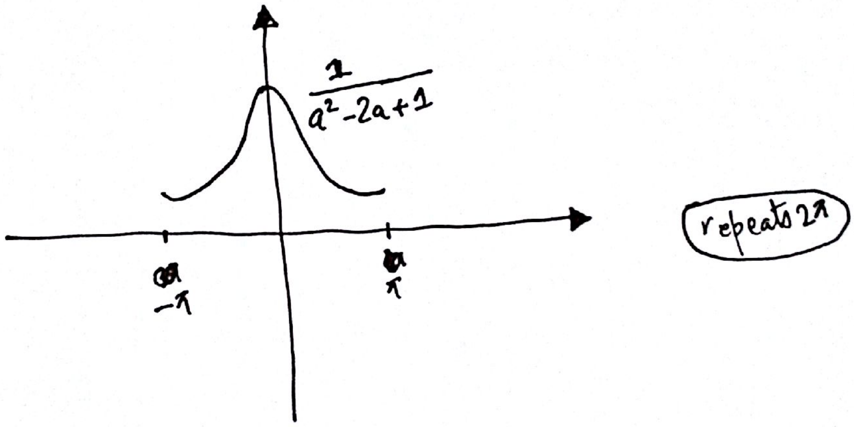
(a) $Re\{X(e^{j\omega})\}$



(b) $Im\{X(e^{j\omega})\}$



(c) $|X(e^{j\omega})|$



(d) $\angle X(e^{j\omega})$